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# Transient invariant and quasi-invariant structures in an example of an aperiodically time dependent fluid flow

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## Abstract

Starting from the concept of invariant KAM tori for nearly-integrable Hamiltonian systems with periodic or quasi-periodic non-autonomous perturbation, the paper analyzes the “analogue” of this class of invariant objects when the dependence on time is aperiodic.

The investigation is carried out in a model motivated by the problem of a travelling wave in a channel over a smooth, quasi- and asymptotically flat (from which the “transient” feature) bathymetry, representing a case in which the described structures play the role of barriers to fluid transport in the phase space.

The paper provides computational evidence for the existence of transient structures also for “large” values of the perturbation size, as a complement to the rigorous results already proven by the first author for real-analytic bathymetry functions.

## 1 Introduction

Since the dawn of the KAM theory, the concept of invariant (KAM) torus has played a key role in the context of nearly-integrable Hamiltonian systems. In the case of systems with one or two degrees of freedom, those invariant structures have the property to “separate” regions of the phase space, obstructing, in this way, phenomena of instability such as the remarkable *drift* of the action variables which can occur in the presence of a larger dimension instead, [Arnol’d(1964)]. Given the intrinsic periodic or quasi-periodic nature of the Celestial Mechanics applications, a little attention has been paid to more general time dependencies. More recently, several studies [Malhotra & Wiggins(1998)] [Wiggins & Mancho(2014)], have suggested that aperiodically time-dependent perturbations could naturally arise in Fluid Dynamics from the Dynamical Systems point of view, and, more specifically, in the *Lagrangian transport* problem. Those studies have underlined the importance of the “aperiodic equivalent” of the concept of KAM torus, as *barrier* to the Lagrangian transport.

Studies on the stability of aperiodic systems have been conducted in the Hamiltonian formalism in [Giorgilli & Zehnder(1992)], [Bounemoura(2013)] and [Fortunati & Wiggins(2014a)]. KAM-type results have been established in [Fortunati & Wiggins(2014b)], [Fortunati & Wiggins(2015)] and [Canadell & de la Llave(2015)].

From a technical point of view, the key difference with respect to the standard perturbative theory lies in the presence of a normalizing canonical transformation, which depends, parametrically, on time, see e.g. [Fortunati & Wiggins(2014b)]. The adjective “transient” is motivated by the

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fact that the perturbation considered in this work is supposed to be decaying in time. We stress that, as remarked in [Fortunati & Wiggins(2016)], despite the (exponential) decay is assumed for convenience, a hypothesis of summability (i.e.  $\int_0^{+\infty} f(I, \varphi, t) dt < +\infty$ , where  $f(I, \varphi, t)$  is the perturbation) turns out to be necessary.

As a consequence, the correspondent normalizing transformation is asymptotic in time to the identity. This suggests that the structure represented by the transformed system can be interpreted, at each time  $t$ , fixed, as an invariant torus. Furthermore, this continuous family approaches, as  $t$  increases, an “asymptotic torus”. It is reasonable to call those objects, *transient invariant* structures.

Despite this concept is already known in the literature, e.g. as *non-autonomous torus*, see [Canadell & de la Llave(2015)], we remark that, however, in the general nearly integrable case described in [Fortunati & Wiggins(2016)] i.e.  $h(I) + \mu f(I, \varphi, t)$ , those structures are not exactly invariant. In fact, a “exponentially small” (in the sense of Nekhoroshev) remainder is present in the normal form. Hence, the unperturbed flow carries a “very small” noise and it is natural to interpret the image of the flow associated to the normalized Hamiltonian as a *quasi-invariant* structure. The correspondent of these objects in the non-aperiodic case are also known as *nearly-invariant* tori, see [Delshams & Gutiérrez(1996)]. However, under the above mentioned hypotheses of time decay, this remainder is harmless as it disappears as  $t \rightarrow +\infty$ , the quasi-invariant structure approaches an asymptotic structure (torus), and the system turns out to be perpetually stable.

A remarkable particular case is discussed in [Fortunati & Wiggins(2016)], when the unperturbed system is isochronous, i.e.  $h(I) = \omega \cdot I$ . In this case, the perturbation can be completely removed via an infinite-stages KAM-type normalization and the transient structures turn out to be exactly invariant.

In any case, the mentioned structures play a crucial role as they act as barriers for the transport in phase space. For a broader panorama on their importance in the Geophysical context we refer to [Samelson & Wiggins(2006)] and references therein.

Instead of choosing an arbitrary aperiodic perturbation, which would impose *ab initio* the required feature to the model, the paper [Fortunati(2018)] suggests a physically relevant case in which such a perturbation naturally arises. More precisely, the case of a travelling wave in a semi-infinite channel, see (2), over a quasi- and asymptotically (in the  $x$  coordinate) flat, smooth bathymetry is considered. In [Fortunati(2018)], the stability of the streamlines in the proximity of the elliptic equilibrium for the unperturbed system is shown in a rigorous way, as a consequence of the results of [Fortunati & Wiggins(2016)]. For an extensive discussion and general results about the problem of the water waves in the presence of a (non-flat) bathymetry we refer to [Craig *et al.*(2012)] and references therein.

The present paper examines a paradigmatic case of the model considered in [Fortunati(2018)] and pushes the investigation beyond the limitation of the rigorous theory (which is known to require extremely small values for the perturbation sizes) by means of numerical methods. Those tools are used to provide evidence of the above described structures and their “transition” in the aperiodic regime.

## 2 Set-up and preliminaries

In the same setting as [Fortunati(2018)], we consider the classical shallow-water equation for an inviscid fluid in the quasi-geostrophic approximation

$$\partial_t(\Delta\psi - F\psi) + \mathcal{F}\partial_x\psi + J(\psi, \delta) = 0, \quad (1)$$

in the semi-infinite channel

$$\mathcal{C} := [0, +\infty) \times \mathbb{T} \ni (x, y), \quad (2)$$

rotating with constant angular velocity  $\Omega$ , being  $\mathcal{F} = \mathcal{F}(\Omega)$ , fixed once and for all. The function  $\psi = \psi(x, y, t)$  is the *streamfunction*,  $F$  is a real parameter and  $\delta = \delta(x, y)$  is the *bathymetry*. As usual,  $J(f_1, f_2) := \partial_x f_1 \partial_y f_2 - \partial_y f_1 \partial_x f_2$  denotes the Jacobian operator.

Given a streamfunction  $\psi$ , the velocity field is associated through the well known equations

$$\dot{x} = -\partial_y \psi \quad ; \quad \dot{y} = \partial_x \psi.$$

The equation (1) appear to be “linearized” with respect to its classical form, see e.g. [Pedlosky(1992)], in the sense that the term  $J(\Delta\psi - F\psi, \psi)$  has been dropped. This assumption is relevant when the amplitude of the studied solution can be regarded as “small”. In any case, as already remarked in [Fortunati(2018)], equation (1) provides itself a non-trivial example for the case of a travelling wave, when the bathymetry is supposed to be more general than periodic in the variable  $x$ . As it is well known, see [Pedlosky(1992)], the travelling wave solution for (1) has the form

$$\Psi(x, y, t) = A \sin(\tilde{m}y) \cos(\kappa x + \sigma t), \quad (3)$$

for all  $\kappa, \tilde{m} \in \mathbb{N}$  and  $A, \mathcal{F}, \sigma \in \mathbb{R}$ , such that the *dispersion relation*

$$\sigma = \kappa \mathcal{F}(\kappa^2 + \tilde{m}^2 + F)^{-1} \quad (4)$$

is satisfied. As for the bathymetry, it will be supposed of the form of a quasi-flat bottom with depth  $d$

$$\delta(x, y) = \mu \tilde{g}(x, y) - d \quad (5)$$

in the hypothesis  $0 < \mu \ll d \ll 1$  where  $\tilde{g}$  represents the deviation of the bathymetry from the flat profile  $\delta(x, y) = -d$ .

Following the natural perturbative interpretation of the parameter  $\mu$ , it is reasonable to substitute the form (5) into (1) and study the convergence of the perturbative series

$$\sum_{j=0}^{+\infty} \mu^j \psi^{(j)}(x, y, t) \quad (6)$$

being  $\psi^{(0)} \equiv \Psi(x, y, t)$  known from (3). This classical approach has been followed in [Fortunati(2018)], showing that, under suitable hypotheses on  $\tilde{g}$ , the series (6) converges for sufficiently small  $\mu$ , see [Fortunati(2018), Proposition 5.1].

In this paper we are interested in the features of the dynamics associated with the truncation of the series (6) at the first order in  $\mu$ , i.e.

$$\psi^{(\leq 1)} = \psi^{(0)} + \mu \psi^{(1)}, \quad (7)$$

when the following paradigmatic form for  $\tilde{g}$  is considered

$$\tilde{g}(x, y) = e^{-\nu x} \sin x \sin(2y). \quad (8)$$

The advantage of the simple structure given by (8) consists in the possibility to choose an ansatz for  $\psi^{(1)}$  which is valid also for  $\nu = 0$ . This represents a substantial difference with respect to the general case in which the two cases  $\nu = 0$  and  $\nu > 0$  are required to be discussed separately. Basically, the algorithm used in [Fortunati(2018)] in order to construct the functions

$\psi^{(j)}$  would require, for this particular model, the hypothesis  $\nu > 0$ . It can be easily seen that, in this example (or possible related generalizations), the exponential term “decouples” from the trigonometric part and the perturbative step can be carried out easily with the same ansatz. In this way, the periodic case is directly interfaced to the aperiodic one simply by changing the value of  $\nu$  from zero to greater than zero. The presence of the coefficient 2, in the term  $\sin(2y)$ , is introduced in order to prevent a symmetry occurring in the limit  $\nu = 0$  which turns out to produce a trivial perturbation.

The study of equation (1) in the channel  $\mathcal{C}$ , clearly requires a set of boundary conditions, i.e.

$$\partial_x \psi(x, 0) \equiv \partial_x \psi(x, 2\pi) \equiv 0, \quad \forall x \in [0, +\infty).$$

It is immediate to see that this imposes analogous conditions on the bathymetry (8). That is why, besides the coefficient 2 justified above, the function (8) is amongst the simplest examples that can be considered for this model.

At the price of a conceptually irrelevant loss of generality but with a great advantage from the notational point of view, we shall choose some particular values for the parameters at hand, as discussed below

**Proposition 2.1.** *Set  $v(z; K_1, K_2) := K_1 \sin z + K_2 \cos z$ . Let us suppose  $\kappa = \tilde{m} = -\mathcal{F} = 1$  and  $F = 8$ . Then the term  $\psi^{(1)}$  reads as, for all  $\nu \geq 0$ ,*

$$\psi^{(1)} = e^{-\nu x} \sum_{i=1,2} [v(x; A_i, B_i) \sin(x - \epsilon t) + v(x; C_i, D_i) \cos(x - \epsilon t)] \cos(iy) \sin((3 - i)y), \quad (9)$$

where  $\epsilon := -\sigma = 1/10$  (according to (4)), and  $A_i, B_i, C_i, D_i$  are functions of  $\nu$  (see Appendix B for their explicit expressions).

*Proof.* Straightforward by substituting the truncated expression (7) with (3) and (9) into (1) then determining the constant  $A_{1,2}, B_{1,2}, C_{1,2}$  and  $D_{1,2}$  by comparing the first order terms in  $\mu$ .  $\square$

### 3 The model: setting and unperturbed system

As anticipated, we shall consider the model obtained by considering  $\psi^{(\leq 1)}$  as (approximated) streamfunction and study the dynamic associated to it.

As usual, see [Knobloch & Weiss(1987)], we first perform the following Galileian transformation

$$\mathcal{G} : (Y, X) \rightarrow (y, x - \epsilon t).$$

In such a way, the unperturbed system is autonomous and the travelling wave is stationary in the new coordinate system. Its equations are

$$\begin{cases} \dot{X} &= -\partial_Y \tilde{\psi}^{(0)} = \epsilon - \cos X \cos Y \\ \dot{Y} &= \partial_X \tilde{\psi}^{(0)} = -\sin X \sin Y \end{cases},$$

where

$$\tilde{\psi}^{(0)} := \psi^{(0)} \circ \mathcal{G} = -\epsilon Y + \sin Y \cos X. \quad (10)$$

It is immediate to check that the following points

$$(X_e, Y_e^\pm) = (0, \pm \arccos \epsilon) \sim (0, \pm 1.4706), \quad (X_h^\pm, Y_h) = (\pm \arccos \epsilon, 0)$$

are equilibria for the system, elliptic and hyperbolic, respectively. The two hyperbolic points are joined by a *separatrix*, described by the implicit equation  $\epsilon Y - \sin(Y) \cos(X) = 0$ . This curve encloses our “region of interest” and contains  $(X_e^\pm, Y_e)$  (we are disregarding the further elliptic equilibria outside this region). More precisely, as anticipated in the introduction, we will concentrate our analysis to the dynamic in the region bounded by the separatrix and the positive  $Y$ -axis, for evidence of stability as extension of the local (but rigorous) result of [Fortunati(2018)], proven in a suitable neighbourhood of  $(X_e^+, Y_e)$ .

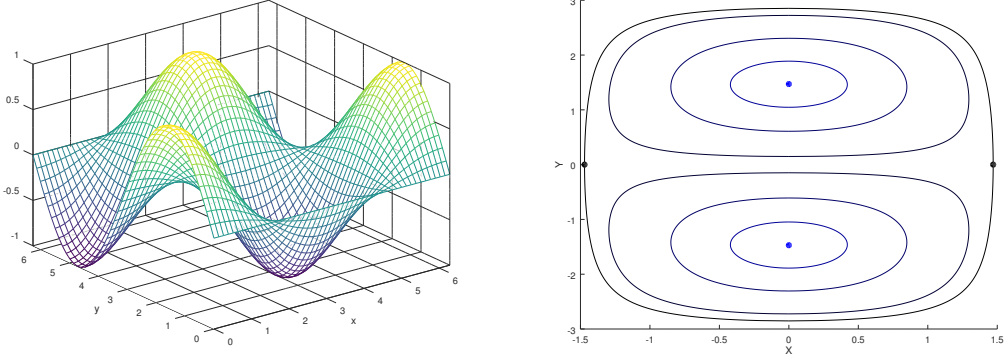


Figure 1: On the left hand side the plot of the (unperturbed) streamfunction  $\psi^{(0)}$  i.e. in the original system of reference for  $t = 0$ . On the right, the phase portrait of the corresponding  $\tilde{\psi}^{(0)}$  in the Galileian system and within the region of interest delimited by the separatrix, here plotted with  $Y \in [-\pi, \pi]$  for simplicity. The hyperbolic and elliptic equilibria are represented by black and blue dots, respectively.

**Remark 3.1.** The left hand side plot in Fig. 1 of  $\psi^{(0)}$  points out its well known geometric interpretation: it represents the actual travelling wave profile along the channel  $\mathcal{C}$ , parallel to the  $x$ -axis. The transformed  $\tilde{\psi}^{(0)}$  has a topologically equivalent structure. Hence, a property of stability around the elliptic points implies that the shape in proximity of the wave crest (or through) e.g. located at  $x = 0$  for  $t = 0$ , is not destroyed for  $t > 0$  by the effect of the bathymetry, but only “deformed”, by an amount “proportional” to  $\mu$ .

## 4 The perturbed system and numerical experiments

When the perturbation is taken into account, one immediately realizes that the transformation  $\mathcal{G}$  introduces an *aperiodic time dependence* due to the non-autonomous term appearing in the exponential. This can be seen as a very embryonic example of a general feature of the system when a non-periodic bathymetry is considered, as pointed out in [Fortunati(2018)].

After a time rescaling  $t \rightarrow \epsilon t$ , the (canonical) equations of the perturbed system are given by

$$\dot{X} = -\partial_Y \tilde{\psi}^{(\leq 1)} \quad ; \quad \dot{Y} = \partial_X \tilde{\psi}^{(\leq 1)} \quad (11)$$

where  $\tilde{\psi}^{(\leq 1)} := \psi^{(\leq 1)} \circ \mathcal{G}$  i.e.

$$\begin{aligned} \tilde{\psi}^{(\leq 1)} &= 10 \cos X \sin Y - Y \\ &+ 10\mu e^{-\nu(X+t)} \sum_{i=1,2} [v(X+t; A_i, B_i) \sin X + v(X+t; C_i, D_i) \cos X] \cos(iY) \sin((3-i)Y). \end{aligned} \quad (12)$$

The numerical experiments presented below have involved the use of a *symplectic integrator*. The features of this class of schemes for the simulation of Hamiltonian systems are well known and we refer to the comprehensive essays of the literature (e.g. [Sanz-Serna(1992)]) for a complete panorama of them. We only mention that the peculiar property of the symplectic methods of *exact* energy conservation, plays a key role in the computation of invariant structures such as tori that would otherwise appear as “noisy” on the Poincaré section, even in the presence of higher order (but non-symplectic) schemes, if run for a sufficiently long time interval. That is why the field of Celestial Mechanics has witnessed a remarkable intervention of those numerical methods. Going more deeply into the implementation aspects, we recall that, as usual, the system (12) can be trivially treated as an autonomous system (i.e. in which the total energy is conserved) simply by introducing a variable, say  $\eta$ , canonically conjugated to the time. In such a way the new Hamiltonian reads as

$$\tilde{\psi}_{\text{aut}}^{(\leq 1)}(X, Y, t, \eta) := \tilde{\psi}^{(\leq 1)}(X, Y, t) + \eta.$$

Given the low dimension of the problem, the *implicit midpoint* scheme

$$z^{(k+1)} = z^{(k)} + hJ\nabla H((z^{(k)} + z^{(k+1)})/2) \quad (13)$$

( $J$  is the fundamental symplectic matrix,  $z = (x, y)$  is the phase space vector and  $h$  is the timestep) turns out to be an appropriate choice for the system at hand. In fact, the Newton method used to solve the system (13) at each step (indexed by  $k$ ), requires a matrix inversion which can be computed explicitly in our case. From this point of view, the Störmer-Verlet method does not offer particular advantages.

#### 4.1 The non-decaying case $\nu = 0$

In this case, the model is simply reduced to a periodically perturbed system. The numerical investigation based on the Poincaré section and reported in Fig. 2, shows the well known phenomenon of preservation of invariant structures (tori) for sufficiently small  $\mu$ , as predicted by the KAM theory.

As  $\mu$  increases, a broader area is filled with non-regular motions and a fluid particle placed in the area enclosed by the unperturbed separatrix (compare with r.h.s panel of Fig. 1) may not be stay trapped between two invariant curves, but may be transported, in principle, arbitrarily far from the initial point. This disordered set is classically formed by the splitting of the heteroclinics and is hyperbolic in nature (see [Malhotra & Wiggins(1998)] for questions related to the Melnikov integral in aperiodically perturbed systems). For “larger” values of  $\mu$ , non-regular motions take place also inside the area enclosed by some surviving invariant curves (Fig. 3).

The value  $\mu \sim 0.2$  (not shown) leads to the almost complete destruction of invariant tori.

#### 4.2 The decaying case $\nu > 0$

In this section we investigate the behaviour of the system in the case of asymptotically flat bathymetry. As a result, the perturbation becomes aperiodic in time. The aim is now to describe Fig. 4 in which some transient structures are shown.

Operationally speaking, it should be said that the continuous phenomenon of transition from the initial structure to the final one is generally quite clear and manifest by looking at the plot **during** the simulation. Unfortunately, the final (static) plot it is not as expressive as the animated one and it can even be mistaken for a plot of a non-regular motion, see Fig. 1.h.s. panel for a “chromatic” animation. That is why we have chosen to plot the “earlier” and the

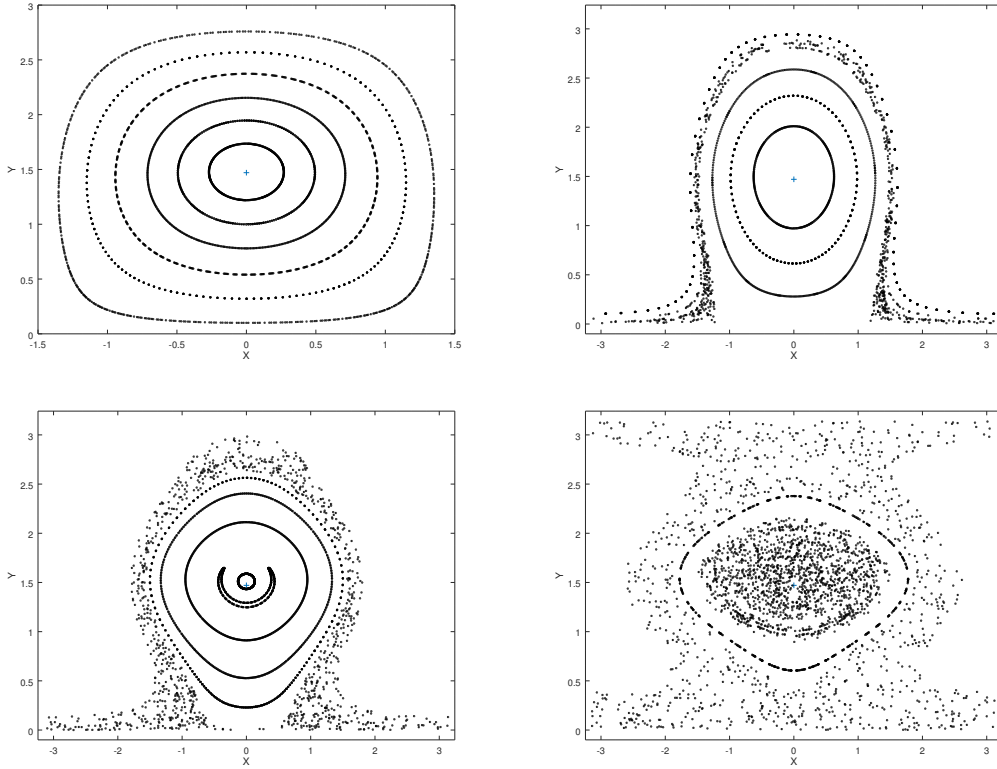


Figure 2: Poincaré section of the perturbed system associated to (12) for  $\mu = 0.005, 0.025, 0.075, 0.175$ , respectively, and  $\nu = 0$  (see Fig. 3 for the case  $\mu = 0.1$ ). In the first panel, qualitatively very similar to the one on the r.h.s. of Fig. 1, the preservation of the (most of the) curves surrounded by the separatrix. In the other panels, the appearance of heteroclinic phenomena, KAM islands, as well as the progressive destruction of invariant tori. The section above corresponds to instants  $t = 2k\pi$ , in which the first 500 points ( $k = 1, \dots, 500$ ) are displayed. The blue cross represents the point  $(X_e, Y_e^+)$ .

“later” stages of the evolution process. More precisely, an initial set of points, plotted in blue, are selected on the Poincaré section in the time interval  $t \in [0, \alpha T_{\text{fin}}]$  (with  $\alpha \in [0.01, 0.05]$ ), while the final set of points, plotted in red, in the interval  $t \in [0.9T_{\text{fin}}, T_{\text{fin}}]$ .

A cautionary word is in order about the Poincaré section. It is well known that the correct definition of the latter is related to the time periodicity of the vector field at hand. In principle, this feature is destroyed by the presence of the aperiodic term. However, in our particular case, we have chosen a very “mild” aperiodicity and the decay acts only as damping of the perturbation, while its frequency it is not affected if  $\nu > 0$ . Furthermore, and this is a general property, the system converges exponentially to an autonomous one, in which the mentioned section is, again, properly defined. On the other hand, it should be said that, the section used is always transverse (as, trivially,  $\dot{t} = 1$ ) it is shown to be, also in the aperiodic case, a valid tool to examine the transition process we are interested in.

As it is natural to expect, the structures in red are a good approximations of a torus in the classical sense (either resonant or non-resonant), which is meant to be the asymptotic torus. The initial structure in blue, has the “resemblance” of a torus but it turns out to be deformed, very quickly, especially for higher values of the perturbation (see Fig. 4, last panel) or higher values



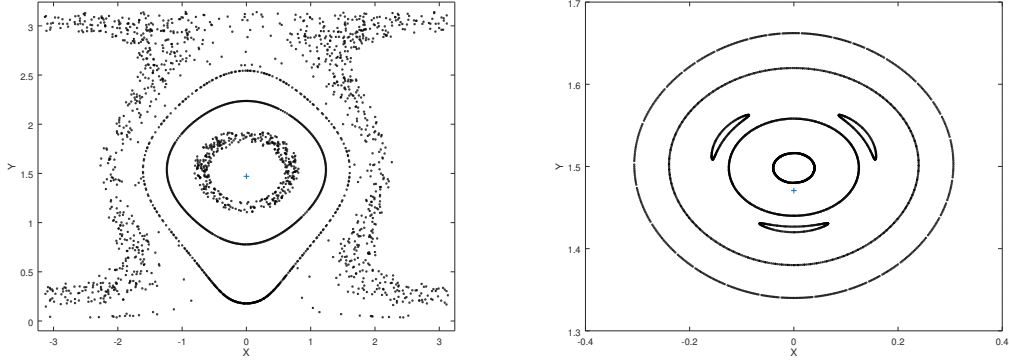


Figure 3: The case  $\mu = 0.1$  and  $\nu = 0$ . On the r.h.s. panel, invariant tori and islands in the area inside the non-regular set shown in the l.h.s. panel.

of  $\nu$  (Fig. 5, r.h.s. panel). In any case it is useful to imagine that the points in blue, describing an “instantaneous torus”, accumulate on the asymptotic structure in red, by spiralling towards it.

## 5 A Perturbation Theory point of view

The aim of this section is to point out, for the concrete example studied in this paper, the key difference between the standard perturbation theory, applicable in the case  $\nu = 0$  and the perturbative approach of [Fortunati & Wiggins(2016)], apt to treat the case  $\nu > 0$  and where the time decay give rise to a phenomenon of *negligibility* of the small divisors effect.

As it is well known from the classical Birkhoff normal form theory, it is possible to construct a local transformation of coordinates

$$\mathcal{T} : (X, Y) \rightarrow (I, \varphi)$$

defined in a neighbourhood of the point, e.g.  $(X_e^+, Y_e)$ , such that the unperturbed streamfunction takes the integrable form

$$H^{(0)} = H^{(0)}(I) := \tilde{\psi}^{(0)} \circ \mathcal{T}.$$

One realizes immediately that the transformation  $\mathcal{T}$  is nothing but the composition of

- a translation of the elliptic point to the origin
- a Birkhoff normal form
- a Poincaré transformation

see e.g. [Iacob et al.(2014)] or [Fortunati(2018)] and Appendix A for more details in this particular case. Let us now state the following

**Proposition 5.1.** *There exists a canonical transformation of variables*

$$\mathcal{T}^{(\leq 2)} : (X, Y) \rightarrow (I, \varphi)$$

such that

$$\Psi^{(\leq 1, 2)} := \psi^{(\leq 1)} \circ \mathcal{T}^{(\leq 2)} = -lI + \left(l + \frac{5\epsilon^2}{3l}\right) \frac{I^2}{4} + \eta + \mu F(I, \varphi, t; \nu) + \mathcal{O}(I^3), \quad (14)$$

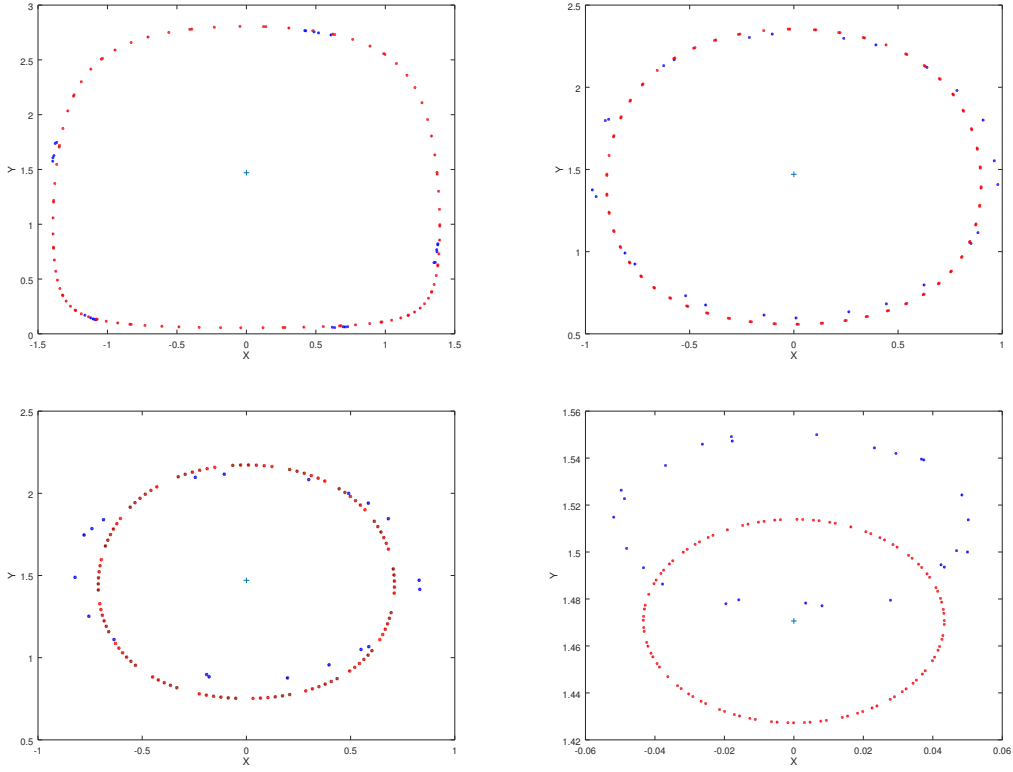


Figure 4: The case  $\nu = 0.0025$  and  $\mu = 0.01, 0.025, 0.05, 0.05$ , respectively. The total number of points computed has been raised to  $k = 1000$  in order to ensure a proper decay of the perturbation, given the small value of  $\nu$ .

where  $l := \sqrt{1 - \epsilon^2}$  (recall that  $\epsilon = 0.1$ ).

*Proof.* The proof is a standard argument of perturbation theory and is given in Appendix A for the sake of completeness, where the transformation  $\mathcal{T}^{(\leq 2)}$  is determined explicitly.  $\square$

As  $\mathcal{T}^{(\leq 2)}$  does not involve the time, it is clear that the system associated to (14) will be periodically or aperiodically perturbed in time if  $\nu = 0$  or  $\nu > 0$ , respectively.

Let us suppose for a moment that the higher orders  $\mathcal{O}(I^3)$  are assumed to be negligible (e.g. if we are “sufficiently” close to the elliptic point). In this case, the frequency of the unperturbed system is

$$\omega(I) = \left( \frac{I}{2} \left( l + \frac{5\epsilon^2}{3l} \right) - l, 1 \right).$$

If  $\nu = 0$ , standard KAM type arguments can be applied, but it is well known that the program fails, at least in general, if  $\omega(I)$  is resonant i.e.  $I \in \mathcal{W}$ , where

$$\mathcal{W} := \{6(l^2 + ln)/(3l^2 + 5\epsilon^2) : n \in \mathbb{Z}\}$$

or “weakly” non-resonant, so that nothing can be said about the perpetual stability of the system. *Au contraire*, if  $\nu > 0$  (no matter how small) the theory from [Fortunati & Wiggins(2016)] can be applied and the following result holds

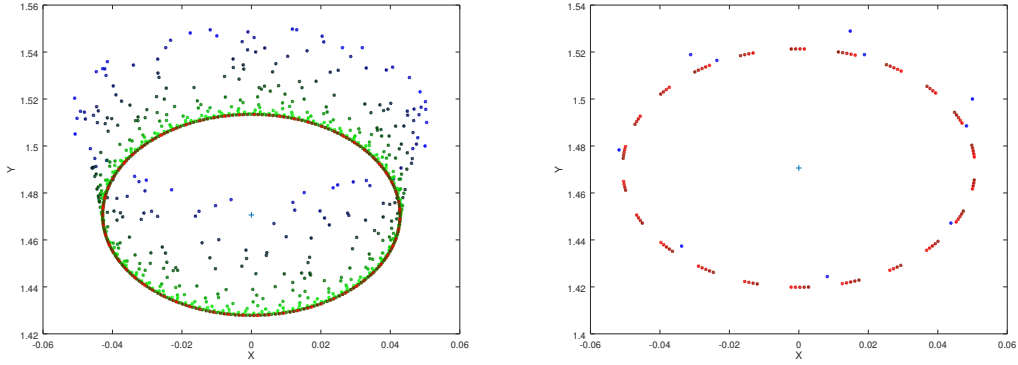


Figure 5: In the l.h.s. panel, simulation with  $\mu = 0.05$  and  $\nu = 0.0015$ . The colour of the points is given by an (empiric) function of the time: predominant blue during the first stage of the simulation, green in the middle and red towards the end. In the r.h.s. panel the “partial” representation (in the sense of Fig. 4) in which  $\mu = 0.015$  and  $\nu = 0.05$  have been chosen: due to the larger value of  $\nu$  the initial structure rapidly accumulates on the asymptotic one.

**Proposition 5.2** ([Fortunati(2018)]). *For all  $I(0)$ ,  $\nu > 0$  and sufficiently small  $\mu$ , the system (14) is perpetually stable.*

Furthermore, the following statement holds

**Proposition 5.3** ([Fortunati & Wiggins(2014b)], [Canadell & de la Llave(2015)]). *The motions associated to (14) with  $\omega(I)$  Diophantine are preserved for sufficiently small  $\mu$ .*

## 6 Conclusions

Numerical experiments conducted on a nearly-integrable Hamiltonian system with an aperiodically time-dependent perturbation have shown evidence of a large set of transient quasi-invariant structures. The model at hand is an approximation of the physically relevant case of a wave, travelling in the channel  $\mathcal{C}$  over a bathymetry with “parametrically driven” decay (and aperiodicity in  $x$ ).

The mentioned transient structures, asymptotic in time to invariant tori, can act as barriers to non-regular motions and ensure the stability of the fluid streamlines in certain regions.

As already stressed by the theory, see [Fortunati & Wiggins(2016)], the structures exist (also) in the presence of **arbitrarily slow** perturbation decay. In the experiments shown in Fig. 4, the size of the perturbation is reduced to  $10\mu \exp(-\nu T_{\text{fin}}) \sim 10^{-8}$  only for  $T_{\text{fin}} = 2000\pi$ . This suggests that the phenomena described in this paper take place for every (smooth) function  $\tilde{g}$ , as long as it decays over a “very large” time interval, emphasizing a certain operative relevance of the described transient structures.

## Acknowledgments

The numerical experiments and the plots reported have been performed with GNU Octave [Eaton *et al.*(2017)] and the algebraic manipulations with Maxima [Maxima(2014)].

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## Appendix A

Let us recall (10) and start with the translation of the elliptic point to the origin

$$\mathcal{R} : (X, Y) = (Q, P + \arccos(\epsilon)).$$

Then we expand the obtained function up to the order four, obtaining

$$\mathcal{H} := \psi^{(\leq 1)} \circ \mathcal{R} = -\frac{l}{2}(Q^2 + P^2) - \frac{\epsilon}{2}(Q^2P + P^3) + \frac{l}{4}\left(\frac{Q^4}{6} + Q^2P^2 + \frac{P^4}{6}\right) + \mathcal{O}_5(P, Q).$$

Now we switch to the usual complex coordinates

$$Q = (\sqrt{2})^{-1}(x + iy) \quad ; \quad P = i(\sqrt{2})^{-1}(x - iy),$$

getting

$$\mathcal{H} = H_2 + H_3 + H_4 + \dots \tag{15}$$

where, in particular

$$\begin{cases} H_2 &:= -ilxy \\ H_3 &:= -(6\sqrt{2})^{-1}\epsilon (ix^3 + 3x^2y - 3ixy^2 + y^3) \\ H_4 &:= -24^{-1}l(x^4 + 6x^2y^2 + y^4) \end{cases}.$$

In order to cast (15) in “normal form”, i.e.  $\mathcal{H} = Z_2 + Z_4 + \mathcal{O}_5(P, Q)$  where  $Z_{2,4}$  are suitable functions, we use the well known *Lie Transform* method, see e.g. [Giorgilli(2003)]. More precisely we write the first three rows of the scheme

$$\begin{cases} Z_2 &= H_2 \\ 0 &= E_1H_2 + H_3 \\ Z_4 &= E_2H_2 + E_1H_3 + H_4 \end{cases},$$

where, once set  $\{F, G\} := \partial_x F \partial_y G - \partial_y F \partial_x G$ , by definition

$$E_s := \begin{cases} \text{Id} & s = 0 \\ \{\chi_1, \cdot\} & s = 1 \\ \frac{1}{s} \sum_{j=1}^s j \mathcal{L}_{\chi_j} E_{s-j} & s \geq 2 \end{cases} \tag{16}$$

being  $\chi := \{\chi_1, \chi_2\}$  is the (unknown) generating sequence.

The solution of the homological equation arising from the second row, i.e.  $\{\chi_1, H_2\} = H_3$ , is easily computed as

$$\chi_1 = (18\sqrt{2}l)^{-1}\epsilon (x^3 + 9ix^2y - 9xy^2 + iy^3).$$

By substituting the latter in the third row of the diagram we get, similarly,

$$\chi_2 = (96l^2)^{-1} [i(\epsilon^2 - l^2)x^4 - 8\epsilon^2xy^3 - 8\epsilon^2x^3y + i(l^2 - \epsilon^2)y^4].$$

The above obtained generating functions read in the original set of variables as

$$\chi_1 = (18l)^{-1}\epsilon Q(5Q^2 + 3P^2), \quad \chi_2 = (48l^2)^{-1}QP [(5\epsilon^2 - l^2)Q^2 + (3\epsilon^2 + l^2)P^2].$$

Those can be used to write the required change of variables via the (truncated) formula,

$$\mathcal{B}^{(\leq 4)} : (Q, P) = \left( \text{Id} + \mathcal{L}_{\chi_1} + \frac{1}{2} \mathcal{L}_{\chi_1}^2 + \mathcal{L}_{\chi_2} \right) (Q', P'),$$

see [Giorgilli(2003), Chapter 4], obtaining

$$\begin{cases} Q &= Q' + \frac{\epsilon}{3l} P' Q' - \left( \frac{5\epsilon^2}{144l^2} + \frac{1}{48} \right) (Q')^3 + \left( \frac{31\epsilon^2}{144l^2} + \frac{1}{16} \right) (P')^2 Q' \\ P &= P' + \frac{\epsilon}{6l} (P')^2 - \frac{5\epsilon}{6l} (Q')^2 + \frac{(9l^2 - 65\epsilon^2)}{144l^2} P' (Q')^2 + \frac{(3l^2 + 5\epsilon^2)}{144l^2} (P')^3 \end{cases} \quad (17)$$

It is immediate to check that

$$\mathcal{H} \circ \mathcal{B}^{(\leq 4)} = -\frac{l}{2} (Q^2 + P^2) + \frac{5\epsilon^2 + 3l^2}{48} (Q^2 + P^2)^2 + \mathcal{O}_5(Q, P)$$

which is the desired normal form up to the order four. In fact, it is now sufficient to perform the classical Poincaré transformation

$$\mathcal{P} : (Q', P') = (\sqrt{2I} \cos \varphi, \sqrt{2I} \sin \varphi),$$

in order to obtain (14). In conclusion,  $\mathcal{T}^{(\leq 2)} := \mathcal{R} \circ \mathcal{B}^{(\leq 4)} \circ \mathcal{P}$ .

## Appendix B

$$\begin{aligned} A_1 &:= -80(\nu^{10} + 170\nu^8 + 9630\nu^6 + 196080\nu^4 + 794409\nu^2 + 53550)/\Delta \\ A_2 &:= 20(\nu^{10} + 135\nu^8 + 5650\nu^6 + 63350\nu^4 - 60211\nu^2 - 123165)/\Delta \\ B_1 &:= -160\nu(\nu^8 + 114\nu^6 + 3452\nu^4 + 20006\nu^2 + 88987)/\Delta \\ B_2 &:= 40\nu(\nu^8 + 44\nu^6 - 1138\nu^4 - 30564\nu^2 - 354503)/\Delta \\ C_1 &:= 10\nu(\nu^{12} + 170\nu^{10} + 9491\nu^8 + 180796\nu^6 + 346327\nu^4 - 1950726\nu^2 + 1315061)/\Delta \\ C_2 &:= 40\nu(5\nu^{10} + 717\nu^8 + 33826\nu^6 + 539978\nu^4 + 1761609\nu^2 + 394585)/\Delta \\ D_1 &:= 10(\nu^{12} + 50\nu^{10} - 4037\nu^8 - 221556\nu^6 - 1714649\nu^4 - 5328734\nu^2 - 390915)/\Delta \\ D_2 &:= 40(15\nu^{10} + 1375\nu^8 + 38598\nu^6 + 52177\nu^4 + 1324347\nu^2 - 20349)/\Delta \end{aligned}$$

where

$$\Delta := \nu^{12} + 230\nu^{10} + 19503\nu^8 + 736212\nu^6 + 11343695\nu^4 + 39298918\nu^2 + 1147041.$$

**Remark 6.1.** Note that  $\lim_{\nu \rightarrow 0} B_{1,2}, C_{1,2} = 0$ .

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